

ON WEIGHTED BIDEGREE OF POLYNOMIAL AUTOMORPHISMS OF  $\mathbb{C}^2$ 

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ABSTRACT. Let  $F = (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be a polynomial automorphism. It is well known that  $\deg F_1 \mid \deg F_2$  or  $\deg F_2 \mid \deg F_1$ . On the other hand, if  $(d_1, d_2) \in \mathbb{N}_+^2 = (\mathbb{N} \setminus \{0\})^2$  is such that  $d_1 \mid d_2$  or  $d_2 \mid d_1$ , then one can construct a polynomial automorphism  $F = (F_1, F_2)$  of  $\mathbb{C}^2$  with  $\deg F_1 = d_1$  and  $\deg F_2 = d_2$ .

Let us fix  $w = (w_1, w_2) \in \mathbb{N}_+^2$  and consider the *weighted degree* on  $\mathbb{C}[x, y]$  with  $\deg_w x = w_1$  and  $\deg_w y = w_2$ . In this note we address the structure of the set  $\{(\deg_w F_1, \deg_w F_2) : (F_1, F_2) \text{ is an automorphism of } \mathbb{C}^2\}$ .

## 1. INTRODUCTION

Let us fix  $n$ -tuple  $w = (w_1, \dots, w_n) \in \mathbb{N}_+^n = (\mathbb{N} \setminus \{0\})^n$ . In this note we will write  $\deg h$  for the usual total degree of a polynomial  $h \in \mathbb{C}[x_1, \dots, x_n]$  and  $\deg_w h$  for the *weighted degree* of  $h$  with respect to  $w$ , where

$$(1) \quad \deg_w h = \max \{ \alpha_1 w_1 + \dots + \alpha_n w_n : c_\alpha \neq 0 \}$$

for

$$(2) \quad h = \sum_{\alpha=(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n} c_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

In other words, we assume that  $\deg_w x_1 = w_1, \dots, \deg_w x_n = w_n$ .

If  $F = (f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a polynomial mapping, then by *multidegree* of  $F$  we mean the following  $n$ -tuple  $\text{mdeg } F = (\deg f_1, \dots, \deg f_n) \in \mathbb{N}^n$  and by the *weighted multidegree* of  $F$  with respect to the *weight*  $w$  we mean the following one  $\text{mdeg}_w F = (\deg_w f_1, \dots, \deg_w f_n) \in \mathbb{N}^n$ . Sometimes, when  $n = 2$ ,  $\text{mdeg } F$  is also called *bidegree* of  $F$  and  $\text{mdeg}_w F$  is called *weighted bidegree* of  $F$ .

Let us recall that a polynomial automorphism  $F$  of  $\mathbb{C}^n$  is called *tame* if it can be obtained as a composition of affine and triangular automorphisms. As usual a mapping  $G = (G_1, \dots, G_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is called *affine* if  $\deg G_i = 1$  for  $i = 1, \dots, n$ , and a mapping  $H = (H_1, \dots, H_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is called *triangular* if for some permutation  $\sigma$  of  $\{1, \dots, n\}$  we have  $H_{\sigma(i)} = c_i \cdot x_{\sigma(i)} + h_i$  for  $i = 1, \dots, n$  and some  $c_i \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ,  $h_i \in \mathbb{C}[x_{\sigma(1)}, \dots, x_{\sigma(i-1)}]$ .

In what follows we will write  $\text{Aut}(\mathbb{C}^n)$  for the group of the all polynomial automorphisms of  $\mathbb{C}^n$  and  $\text{Tame}(\mathbb{C}^n)$  for the subgroup of  $\text{Aut}(\mathbb{C}^n)$  containing all the tame automorphisms. Then, one can consider two functions (also denoted  $\text{mdeg}$  and  $\text{mdeg}_w$ ) mapping  $\text{Aut}(\mathbb{C}^n)$  into  $\mathbb{N}_+^n$ . It is well-known [3, 11] that

$$(3) \quad \text{mdeg}(\text{Aut}(\mathbb{C}^2)) = \text{mdeg}(\text{Tame}(\mathbb{C}^2)) = \{(d_1, d_2) \in \mathbb{N}_+^2 : d_1 \mid d_2 \text{ or } d_2 \mid d_1\}.$$

Since  $\text{Aut}(\mathbb{C}^2) = \text{Tame}(\mathbb{C}^2)$ , we obviously have

$$(4) \quad \text{mdeg}_w(\text{Aut}(\mathbb{C}^2)) = \text{mdeg}_w(\text{Tame}(\mathbb{C}^2)).$$

This note address the structure of the above set. Namely we show the following

**Theorem 1.1.** *Let  $w = (w_1, w_2) \in \mathbb{N}_+^2$ . Then the set  $\text{mdeg}_w(\text{Aut}(\mathbb{C}^2))$  is equal to*

$$\begin{aligned} & \{(d_1, d_2) \in (w_1 \mathbb{N}_+)^2 : d_1 \mid d_2 \text{ or } d_2 \mid d_1, \max\{d_1, d_2\} \geq \tilde{w}, \min\{d_1, d_2\} < \tilde{w} \Rightarrow \min\{d_1, d_2\} = \underline{w}\} \\ & \cup \{(d_1, d_2) \in (w_2 \mathbb{N}_+)^2 : d_1 \mid d_2 \text{ or } d_2 \mid d_1, \max\{d_1, d_2\} \geq \tilde{w}, \min\{d_1, d_2\} < \tilde{w} \Rightarrow \min\{d_1, d_2\} = \underline{w}\} \\ & \cup \{(w_1, w_2), (w_2, w_1), (\tilde{w}, \tilde{w})\}, \end{aligned}$$

where  $\tilde{w} := \max\{w_1, w_2\}$  and  $\underline{w} := \min\{w_1, w_2\}$ .

Notice that if  $w_1 = w_2$ , then the set given on the right-hand side of the above equality is equal to  $\{(d_1, d_2) \in (w_1 \mathbb{N}_+)^2 : d_1 | d_2 \text{ or } d_2 | d_1\}$ . In particular, for  $(w_1, w_2) = (1, 1)$ , one obtain the equality (3).

For information about multidegrees of tame and wild automorphisms of  $\mathbb{C}^3$  see [2, 4, 5, 6, 7, 8, 9, 10, 12].

## 2. LENGHT OF $F \in \text{AUT}(\mathbb{C}^2)$ AND THE WEIGHTED BIDEGREE

In this section we show that  $\text{mdeg}_w(\text{Aut}(\mathbb{C}^2))$  is contained in the set given on the right-hand side of the equality of Theorem 1.1. More precisely we show Theorems 2.3 and 2.4 below, but we start with the following

**Proposition 2.1** (see e.g. [7, Prop. 9.2]). *If  $F \in \text{Aut}(\mathbb{C}^2)$ , then there is a number  $l \in \mathbb{N}$  (possibly zero), affine automorphisms  $L_1, L_2$  of  $\mathbb{C}^2$  and triangular automorphisms  $T_1, \dots, T_l$  of the forms*

$$(5) \quad T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_i(x)) \in \mathbb{C}^2 \quad \text{for } i = 1, 3, \dots,$$

$$(6) \quad T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_i(y), y) \in \mathbb{C}^2 \quad \text{for } i = 2, 4, \dots,$$

with  $\deg f_i > 1$ , such that

$$F = L_2 \circ T_l \circ \dots \circ T_1 \circ L_1.$$

Moreover, the number  $l$  is unique, and one can require that  $T_i$ ,  $i = 1, \dots, l$ , are of the form (5) for even  $i$  and of the form (6) for odd  $i$ .

**Definition 2.2** (see e.g. [1, p.612]). *Let  $F \in \text{Aut}(\mathbb{C}^2)$  be a polynomial automorphism. The number  $l$  from Proposition 2.1 is called the length of  $F$  and denoted  $\text{length } F$ .*

Now, we are in a position to prove Theorems 2.3 and 2.4.

**Theorem 2.3.** *Let us fix  $w = (w_1, w_2) \in \mathbb{N}_+^2$ . If  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is a polynomial automorphism with  $\text{length } F \leq 1$ , then the weighted multidegree  $\text{mdeg}_w F$  is an element of the following set*

$$\begin{aligned} & \{(w_1, kw_1), (kw_1, w_1), (kw_1, kw_1) : k \in \mathbb{N}_+ \text{ and } kw_1 \geq w_2\} \\ \cup & \{(w_2, kw_2), (kw_2, w_2), (kw_2, kw_2) : k \in \mathbb{N}_+ \text{ and } kw_2 \geq w_1\} \\ \cup & \{(w_1, w_2), (w_2, w_1), (\tilde{w}, \tilde{w})\}, \end{aligned}$$

where  $\tilde{w} := \max\{w_1, w_2\}$ .

**Proof.** If  $\text{length } F = 0$ , then  $F$  is affine and so one can easy check that  $\text{mdeg}_w F$  belongs to  $\{(w_1, w_2), (w_2, w_1), (\tilde{w}, \tilde{w})\}$ .

Assume that  $\text{length } F = 1$ . By Proposition 2.1 we can assume that  $F = L_2 \circ T \circ L_1$ , where  $L_1, L_2$  are affine automorphisms and  $T$  is of the form

$$(7) \quad T : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f(x)) \in \mathbb{C}^2,$$

with  $\deg f > 1$ .

We have three cases: (I)  $\text{mdeg}_w L_1 = (w_1, w_2)$ , (II)  $\text{mdeg}_w L_1 = (w_2, w_1)$  and (III)  $\text{mdeg}_w L_1 = (\tilde{w}, \tilde{w})$ . Thus we have

$$(8) \quad (k_1, k_2) := \text{mdeg}_w (T \circ L_1) = \begin{cases} (w_1, \max\{w_1 \cdot \deg f, w_2\}), & \text{for case (I),} \\ (w_2, \max\{w_2 \cdot \deg f, w_1\}), & \text{for case (II),} \\ (\tilde{w}, \tilde{w} \cdot \deg f), & \text{for case (III).} \end{cases}$$

Since  $L_2$  is affine, it follows that  $(d_1, d_2) := \text{mdeg}_w F = \text{mdeg}_w (L_1 \circ T \circ L_1)$  belongs to  $\{(k_1, k_2), (k_2, k_1), (\tilde{k}, \tilde{k})\}$ , where  $\tilde{k} := \max\{k_1, k_2\}$ . Thus, we have:

CASE (I). If  $\max\{w_1 \cdot \deg f, w_2\} = w_2$ , then  $(k_1, k_2) = (w_1, w_2)$  and so  $(d_1, d_2)$  belongs to  $\{(w_1, w_2), (w_2, w_1), (w_2, w_2)\}$  else  $(k_1, k_2) = (w_1, w_1 \cdot \deg f)$  and so  $(d_1, d_2)$  belongs to  $\{(w_1, w_1 \cdot \deg f), (w_1 \cdot \deg f, w_1), (w_1 \cdot \deg f, w_1 \cdot \deg f)\}$ .

CASE (II). If  $\max\{w_2 \cdot \deg f, w_1\} = w_1$ , then  $(k_1, k_2) = (w_2, w_1)$  and so  $(d_1, d_2)$  belongs to  $\{(w_1, w_2), (w_2, w_1), (w_1, w_1)\}$  else  $(k_1, k_2) = (w_2, w_2 \cdot \deg f)$  and so  $(d_1, d_2)$  belongs to  $\{(w_2, w_2 \cdot \deg f), (w_2 \cdot \deg f, w_2), (w_2 \cdot \deg f, w_2 \cdot \deg f)\}$ .

CASE (III). If  $\tilde{w} = w_1$ , then  $(k_1, k_2) = (w_1, w_1 \cdot \deg f)$  and so  $(d_1, d_2)$  belongs to  $\{(w_1, w_1 \cdot \deg f), (w_1 \cdot \deg f, w_1), (w_1 \cdot \deg f, w_1 \cdot \deg f)\}$  else  $\tilde{w} = w_2$ ,  $(k_1, k_2) = (w_2, w_2 \cdot \deg f)$  and so  $(d_1, d_2)$  belongs to  $\{(w_2, w_2 \cdot \deg f), (w_2 \cdot \deg f, w_2), (w_2 \cdot \deg f, w_2 \cdot \deg f)\}$ .

Thus the result follows.  $\square$

**Theorem 2.4.** *Let us fix  $w = (w_1, w_2) \in \mathbb{N}_+^2$ . If  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is a polynomial automorphism with  $\text{length } F \geq 2$ , then the weighted multidegree  $\text{mdeg}_w F$  is an element of the following set*

$$\begin{aligned} & \{(d_1, d_2) \in (w_1 \mathbb{N}_+)^2 : \min\{d_1, d_2\} \geq \max\{w_1, w_2\} \text{ and } (d_1 | d_2 \text{ or } d_2 | d_1)\} \\ \cup & \{(d_1, d_2) \in (w_2 \mathbb{N}_+)^2 : \min\{d_1, d_2\} \geq \max\{w_1, w_2\} \text{ and } (d_1 | d_2 \text{ or } d_2 | d_1)\}. \end{aligned}$$

In particular  $|\text{mdeg}_w F| > |w|$  when  $\text{length } F \geq 2$ .

**Proof.** Let  $l := \text{length } F$ . By Proposition 2.1 we can assume that

$$F = L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1,$$

where  $L_1, L_2$  are affine automorphism of  $\mathbb{C}^2$  and  $T_1, \dots, T_l$  are triangular automorphisms of the forms

$$(9) \quad T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_i(x)) \in \mathbb{C}^2 \quad \text{for } i = 1, 3, \dots,$$

$$(10) \quad T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_i(y), y) \in \mathbb{C}^2 \quad \text{for } i = 2, 4, \dots$$

Let  $(k_1, k_2) := \text{mdeg}_w(T_1 \circ L_1)$ . By (8), we have that  $k_2 > k_1$ ,  $k_2 \geq \max\{w_1, w_2\}$  and  $k_2 \in w_1 \mathbb{N}_+ \cup w_2 \mathbb{N}_+$ . It is easy to see that

$$(11) \quad \text{mdeg}_w(T_2 \circ T_1 \circ L_1) = (k_2 \cdot \deg f_2, k_2)$$

and for  $l > 2$

$$(12) \quad \text{mdeg}_w(T_l \circ \cdots \circ T_1 \circ L_1) = \begin{cases} (k_2 \prod_{i=2}^l \deg f_i, k_2 \prod_{i=2}^{l-1} \deg f_i), & \text{for even } l, \\ (k_2 \prod_{i=2}^{l-1} \deg f_i, k_2 \prod_{i=2}^l \deg f_i), & \text{for odd } l. \end{cases}$$

Thus

$$(13) \quad \text{mdeg}_w(L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1) \in \{(m_1, m_2), (m_2, m_1), (m_2, m_2)\},$$

where  $m_1 := k_2 \prod_{i=2}^{l-1} \deg f_i$  and  $m_2 := k_2 \prod_{i=2}^l \deg f_i$  for  $l > 2$ , and  $m_1 := k_2$  and  $m_2 := k_2 \deg f_2$  for  $l = 2$ . Hence, the result follows.  $\square$

### 3. EXAMPLES

Let  $Z$  denotes the set given in Theorem 1.1. By Theorems 2.3 and 2.4, in order to proof Theorem 1.1, it is enough to show an example of automorphism  $F \in \text{Aut}(\mathbb{C}^2)$  with  $\text{mdeg}_w F = (d_1, d_2)$  for each  $(d_1, d_2) \in Z$ .

Without lose of generality we can assume that  $w_1 \leq w_2$ . First consider the case  $w_1 = w_2$ . Take any  $(d_1, d_2) \in Z$ . Since  $w_2 | d_1$  and  $w_2 | d_2$ , it follows that one can take

$$(14) \quad F = \begin{cases} T_2 \circ T_1, & \text{for } d_1 < d_2, \\ \tilde{T}_2 \circ \tilde{T}_1, & \text{for } d_1 > d_2, \\ L \circ T_1, & \text{for } d_1 = d_2, \end{cases}$$

where  $T_1(x, y) = (x + y^{\frac{d_1}{w_2}}, y)$ ,  $T_2(x, y) = (x, y + x^{\frac{d_2}{w_2}})$ ,  $\tilde{T}_1(x, y) = (x, y + x^{\frac{d_2}{w_2}})$ ,  $\tilde{T}_2(x, y) = (x + y^{\frac{d_2}{d_1}}, y)$  and  $L(x, y) = (x, y + x)$ .

Now, consider the case  $w_1 < w_2$  and take any  $(d_1, d_2) \in Z$ . If  $(d_1, d_2) \in (w_2 \mathbb{N}_+)^2$ , then one can take

$$(15) \quad F = \begin{cases} T_2 \circ T_1, & \text{for } d_1 < d_2, \\ \tilde{T}_2 \circ \tilde{T}_1 \circ \tilde{L}, & \text{for } d_1 > d_2, \\ L \circ T_1, & \text{for } d_1 = d_2, \end{cases}$$

where  $T_1, T_2, \tilde{T}_1, \tilde{T}_2$  and  $L$  are defined as in the case  $w_1 = w_2$ , and  $\tilde{L}(x, y) = (y, x)$ .

If  $(d_1, d_2) \in (w_1 \mathbb{N}_+)^2$ , then we have two cases: (I)  $\min\{d_1, d_2\} \geq w_2$  and (II)  $\min\{d_1, d_2\} = w_1$ . In case (I) one can take

$$(16) \quad F = \begin{cases} T_2 \circ T_1, & \text{for } d_1 < d_2, \\ \tilde{T}_2 \circ \tilde{T}_1, & \text{for } d_1 > d_2, \\ L \circ T_1, & \text{for } d_1 = d_2, \end{cases}$$

where  $T_1(x, y) = (x, y + x^{\frac{d_1}{w_1}})$ ,  $T_2(x, y) = (x + y^{\frac{d_2}{w_2}}, y)$ ,  $\tilde{T}_1(x, y) = (x, y + x^{\frac{d_2}{w_1}})$ ,  $\tilde{T}_2(x, y) = (x + y^{\frac{d_1}{w_2}}, y)$  and  $L(x, y) = (x + y, y)$ .

And, in case (II), we one can take

$$(17) \quad F = \begin{cases} T_1, & \text{for } d_1 < d_2, \\ \tilde{L} \circ T_2, & \text{for } d_1 > d_2, \\ L \circ T_1, & \text{for } d_1 = d_2, \end{cases}$$

where  $T_1(x, y) = (x, y + x^{\frac{d_2}{w_1}})$ ,  $T_2(x, y) = (x, y + x^{\frac{d_1}{w_1}})$ ,  $L(x, y) = (x + y, y)$  and  $\tilde{L}(x, y) = (y, x)$ .

Finally, if  $(d_1, d_2) \in \{(w_1, w_2), (w_2, w_1), (w_2, w_2)\}$ , then one can take

$$(18) \quad f(x, y) = \begin{cases} (x, y), & \text{for } (d_1, d_2) = (w_1, w_2), \\ (y, x), & \text{for } (d_1, d_2) = (w_2, w_1), \\ (x + y, x) & \text{for } (d_1, d_2) = (w_2, w_2). \end{cases}$$

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